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COMPACTNESS FOR A CLASS OF HIT-AND-MISS HYPERSPACES

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In the study of some kind of generalized Vietoris-type topologies for the hyperspace of all nonempty closed subsets of a topological space (X, τ), namely the so called Δ-hit-and-miss-topologies with $\Delta \subseteq Cl(X)$ (or Δ-topologies), which was initiated by the second author in 1965, it is obvious, that the non-compactness of such a hyperspace often depends on the non-compactness even in the lower-semifinite topology (induced by the "hit-sets"), which is contained in all hypertopologies of this type. Otherwise, compactness for these topologies is easily obtained from the compactness of (X, τ) by well-known theorems, if the "miss-sets" are induced either by compact or closed subsets. To obtain a similar result for topologies with "miss-sets" generated by subsets with a property which generalizes both, closedness and compactness especially in the non-Hausdorff case, we use consequently a quite set-theoretical lemma, stated at the beginning.

Let (X, τ) be a topological space. By $\mathfrak{P}(X)$, $\mathfrak{P}_0(X)$, 2^X, $Cl(X)$ and $K(X)$ respectively we denote the power set, the power set without the empty set $\emptyset$, the family of all closed subsets, the family of all nonempty closed subsets and the set of all compact subsets of X. For $B \in \mathfrak{P}(X)$ we define $B^- := \{A \in Cl(X) | A \cap B \neq \emptyset\}$ (hit-set) and $B^+ := \{A \in Cl(X) | A \cap B = \emptyset\}$ (miss-set). (This notation for a miss-set is a deviation from the otherwise used $(X \setminus B)^+$, but in [5], [6] the corresponding hit-and-miss hyperspace topologies also were defined in this manner.) By τ_l we denote the topology for $Cl(X)$, generated by the subbase of all G^-, $G \in \tau$. Now consider $\emptyset \neq \alpha \subseteq \mathfrak{P}(X)$; by τ_α we denote the topology for $Cl(X)$ which is generated from the subbase of all B^+, $B \in \alpha$ and G^-, $G \in \tau$. Of course, for every possible α we have $\tau_l \subseteq \tau_\alpha$; for $\alpha = Cl(X)$ we get the Vietoris topology and for $\alpha = K(X)$ we get the Fell topology for $Cl(X)$. If $\alpha = \Delta \subseteq Cl(X)$, τ_α is called Δ-topology by Beer and Tamaki [2].

Δ-topologies are now studied by several authors (see [2], [3] and the papers quoted in these papers). Δ-topologies were first considered in a paper [6] of the second author. In this paper and in the preceding paper [5] separation axioms, especially Hausdorffness and regularity, for general hit-and-miss topologies and for Δ-topologies were studied. The aim of the present paper is to look for a special class of hit-and-miss topologies τ_α, including Δ-topologies, when $(Cl(X), \tau_\alpha)$ is compact. At first we will extend the notions of the hit-and miss-sets in order to state some quite general lemmas.

By $\Phi(X)$ and $\Phi_0(X)$ we denote the set of all filters and ultrafilters, respectively, on a set X (a filter is not allowed to contain the empty set $\emptyset$); the symbol $\Phi(\varphi)$ (resp. $\Phi_0(\varphi)$) means the set of all filters (resp. ultrafilters) which contain a given filter φ; $\dot{x}$ is the filter generated by a singleton $\{x\}, x \in X$. The symbol q_τ denotes the convergence structure induced by a topology τ, i.e. $q_\tau := \{(\varphi, x) \in \Phi(X) \times X | \varphi \supseteq \dot{x} \cap \tau\}$, so q_τ is a relation between filters and points of a set X.

DEFINITION 1. *For a set X assume $\mathfrak{A} \subseteq \mathfrak{P}(X)$, $M \subseteq X$. Then we call*

$$M^{-\mathfrak{A}} := \{A \in \mathfrak{A} | A \cap M \neq \emptyset\}$$

the M-hit-set w.r.t. $\mathfrak{A}$ and

$$M^{+\mathfrak{A}} := \{A \in \mathfrak{A} | A \cap M = \emptyset\} (= \mathfrak{A} \setminus M^{-\mathfrak{A}} =: (M^{-\mathfrak{A}})^c)$$

the M-miss-set w.r.t. $\mathfrak{A}$.

Obviously we get the usual notions mentioned above with $\mathfrak{A} = Cl(X)$.

If X is a set, $\tau, \mathfrak{A}$ are subsets of $\mathfrak{P}(X)$, then we call $\mathfrak{A}$ *weak complementary w.r.t τ, iff* for every collection $\{G_i | i \in I\} \subseteq \tau$ and every subset $\emptyset \neq R \subseteq X \setminus \bigcup_{i \in I} G_i$ there exists an element A of $\mathfrak{A}$ with $A \subseteq X \setminus \bigcup_{i \in I} G_i$ and $A \cap R \neq \emptyset$. (Equivalently, one may write $\bigcup\{A|\ A \in \mathfrak{P}(X \setminus \bigcup_{i \in I} G_i) \cap \mathfrak{A}\} = X \setminus \bigcup_{i \in I} G_i$ for this.)

Now, we can state a quite set-theoretical, not even surprising, but fairly useful lemma:

LEMMA 2. *Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds*

$$\bigcup_{i \in I} G_i \supseteq K \implies \bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

for every collection $G_i, i \in I, G_i \in \tau$.

If $\mathfrak{A}$ is weak complementary w.r.t. τ, then for every collection $G_i, i \in I, G_i \in \tau$ the implication

$$\bigcup_{i \in I} G_i \supseteq K \impliedby \bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

holds, too.

Proof. Let $\bigcup_{i \in I} G_i \supseteq K$. $A \in K^{-\mathfrak{A}} \Rightarrow A \cap K \neq \emptyset \Rightarrow \emptyset \neq A \cap \bigcup_{i \in I} G_i \Rightarrow \exists i_0 \in I : A \cap G_{i_0} \neq \emptyset \Rightarrow A \in G_{i_0}^{-\mathfrak{A}} \Rightarrow A \in \bigcup_{i \in I} G_i^{-\mathfrak{A}}$.

Conversely, let $\mathfrak{A}$ be weak complementary w.r.t. τ and $\bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$. Assume $\bigcup_{i \in I} G_i \not\supseteq K$. Then $X \setminus \bigcup_{i \in I} G_i \supseteq K \setminus \bigcup_{i \in I} G_i \neq \emptyset$ holds, so there is an $A \in \mathfrak{A}$, $A \subseteq X \setminus \bigcup_{i \in I} G_i$ with $A \cap K \setminus \bigcup_{i \in I} G_i \neq \emptyset$. Thus $A \in K^{-\mathfrak{A}}$, implying $A \in \bigcup_{i \in I} G_i^{-\mathfrak{A}}$. This yields $\exists i_0 \in I : A \cap G_{i_0} \neq \emptyset$ in contradiction to the construction of A. □

COROLLARY 3. *Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds*

$$\bigcup_{i \in I} G_i \supseteq K \iff \bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}} \tag{1}$$

for every collection $G_i, i \in I, G_i \in \tau$ if and only if $\mathfrak{A}$ is weak complementary w.r.t. τ.

Proof. We only have to show, that $\mathfrak{A}$ is weak complementary w.r.t. τ, if (1) holds. Assume, $\mathfrak{A}$ is not weak complementary w.r.t. τ. Then there must be a collection $\{G_i | i \in I\} \subseteq \tau$, such that

$$\bigcup \left\{ A | A \in \mathfrak{P}\left(X \setminus \bigcup_{i \in I} G_i \right) \cap \mathfrak{A} \right\} \not\supseteq X \setminus \bigcup_{i \in I} G_i.$$

Now, we chose

$$K := \left(X \setminus \bigcup_{i\in I} G_i \right) \setminus \bigcup \left\{ A | A \in \mathfrak{P} \left(X \setminus \bigcup_{i\in I} G_i \right) \cap \mathfrak{A} \right\} \neq \emptyset.$$

Then no element of $\mathfrak{A}$, which meets K, can be contained in $X \setminus \bigcup_{i\in I} G_i$, i.e. every element of $K^{-\mathfrak{A}}$ meets $\bigcup_{i\in I} G_i$, too. So, it must meet a G_{i_0}, $i_0 \in I$ and consequently it is contained in $\bigcup_{i\in I} G_i^{-\mathfrak{A}}$. But, by construction, the collection $\{G_i | i \in I\}$ doesn't cover K, so (1) would fail. □

Obviously, if for every collection $\{G_i | i \in I\} \subseteq \tau$ the complement $X \setminus \bigcup_{i\in I} G_i$ itself belongs to $\mathfrak{A}$, or if all singletons $\{x\}, x \in X$ are elements of $\mathfrak{A}$, then $\mathfrak{A}$ is weak complementary w.r.t. τ. So, if τ is a topology on X, $Cl(X)$ and $K(X)$ are weak complementary w.r.t. τ.

COROLLARY 4. *Let (X, τ) be a topological space, $K \subseteq X$ and $\forall i \in I$: $G_i \in \tau$. Then holds*

$$\bigcup_{i\in I} G_i \supseteq K \iff \bigcup_{i\in I} G_i^- \supseteq K^-$$

DEFINITION 5. *Let (X, τ) be a topological space. A subset $A \subseteq X$ is called* weak relative complete *in X, iff*

$$\forall \varphi \in \Phi(A) \cap q_\tau^{-1}(X) : \Phi(\varphi) \cap q_\tau^{-1}(A) \neq \emptyset \, ,$$

i.e. every filter φ on A, which converges in X, has a refinement, converging in A.

PROPOSITION 6. *Let (X, τ) be a topological space and $A \subseteq X$. Then holds:*

(a) A is weak relative complete in X, iff $\Phi_0(A) \cap q_\tau^{-1}(X) = \Phi_0(A) \cap q_\tau^{-1}(A)$, i.e. every ultrafilter on A, which converges in X, converges in A, too.
(b) If A is closed in X, then A is weak relative complete in X.
(c) If A is compact, then A is weak relative complete in X.

(d) If (X, τ) is compact and A is weak relative complete in X, then A is compact, too.

(e) If (X, τ) is Hausdorff, then every weak relative complete subset $A \subseteq X$ is closed in (X, τ).

(f) A is compact iff A is weak relative complete and relative compact.

(g) Weak relative completeness is transitive, i.e. for all $A \subseteq B \subseteq X$ with B weak relative complete in (X, τ) and A weak relative complete in $(B, \tau_{|B})$, the subset A is weak relative complete in (X, τ), too.

Proof:

(a): If A is weak relative complete in X, the assertion about the ultrafilters on A follows immediately from the fact, that an ultrafilter has no proper refinement. Conversely, if a filter φ on A is given, which converges in X, then every refining ultrafilter $\psi \supseteq \varphi$ converges in X, too. Now, by $\Phi_0(A) \cap q_\tau^{-1}(X) = \Phi_0(A) \cap q_\tau^{-1}(A)$, ψ converges in A and is a refinement of φ. So, A is weak relative complete in X.

(b): If A is closed in X, then every point of X, to which a filter on A may converge, belongs to A.

(c): If A is compact, then every ultrafilter on A converges in A and the weak relative completeness of A in X follows from (a).

(d): X compact $\Rightarrow \Phi_0(X) \cap q_\tau^{-1}(X) = \Phi_0(X) \Rightarrow \Phi_0(A) \cap q_\tau^{-1}(X) = \Phi_0(A)$ and by the weak relative compactness of A with (a) we get $\Phi_0(A) \cap q_\tau^{-1}(A) = \Phi_0(A)$, i.e. A is compact.

(e): If A is weak relative complete in (X, τ) and there is a filter $\varphi \in \Phi(A)$, converging to a point $x \in X$. Then there must exist a refining filter $\psi \in \Phi(\varphi)$ which converges to a point $a \in A$. But this filter converges to x, too, because of it's subfilter φ, so by Hausdorffness $x = a \in A$ follows. So, A is closed in (X, τ).

(f): A compact subset A is clearly relative compact, and it is weak relative complete by (c). If A is relative compact[1], then every ultrafilter on A converges in X and so it converges in A by (a) if A is in weak relative complete in addition.

[1] Note, that we call a subset A of a topological space (X, τ) to be *relative compact in X*, iff all ultrafilters on A converge in X. This doesn't imply the compactness of the closure of A, in distinction to another practices to use the notion "relative compact". For more explanation see [1], [7]

(g): Follows immediately from (a), because an ultrafilter on A is an ultrafilter on B, too. So, if it converges in X, it must converge in B and so in A, too, because of the weak relative completeness, successively. □

Obviously, we can define the well-known hyper-topology τ_l not only for the space of the closed subsets, but for any subset $\mathfrak{A}$ of $\mathfrak{P}(X)$: denote by $\tau_{l,\mathfrak{A}}$ the topology for $\mathfrak{A}$, which is generated by the subbase consisting of all $G^{-\mathfrak{A}}$, $G \in \tau$. (This means, we require the images of the open sets of the base space under the mapping ${}^{-\mathfrak{A}} : \mathfrak{P}(X) \to \mathfrak{P}(\mathfrak{A})$ to be open in the hyperspace.) In the same manner we can extend the definition of τ_α for any $\alpha \subseteq \mathfrak{P}(X)$. (This means, we require the images of the members of α under the ${}^{-\mathfrak{A}}$-mapping to be closed in the constructed hyperspace. So, the study of hit-and-miss-hyperspaces is essentially the study of the ${}^{-\mathfrak{A}}$-mapping.)

LEMMA 7. *Let (X, τ) be a topological space, $\alpha \subseteq \mathfrak{P}(X)$ and a collection $\mathfrak{A} \subseteq \mathfrak{P}(X)$ which is weak complementary w.r.t. τ. If (X, τ) is not compact, then $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\emptyset\}$ is not compact in τ_l and consequently not in τ_α, too.*

Proof. If $\mathfrak{A}$ is weak complementary w.r.t. τ, then $\mathfrak{A}_0$ is, too. So, lemma 3 is applicable.

Because X is not compact, there is an open cover $(G_i)_{i\in I}$ of X, such that no finite subfamily covers X. Applying Lemma 2 with $K = X$ and consequently $K^{-\mathfrak{A}_0} = \mathfrak{A}_0$ we find, that $(G_i^{-\mathfrak{A}_0})_{i\in I}$ is an open cover of $\mathfrak{A}_0$. For any subfamily of $(G_i^{-\mathfrak{A}_0})_{i\in I}$ which covers $\mathfrak{A}_0$, the corresponding subfamily of $(G_i)_{i\in I}$ covers X by lemma 2. So, there exists no finite subcover of $\mathfrak{A}_0$ in $(G_i^{-\mathfrak{A}_0})_{i\in I}$. □

Now, we can state the main result.

THEOREM 8. *If (X, τ) is a topological space, then let $\alpha \subseteq \mathfrak{P}(X)$ consist of weakly relative complete subsets of X. Then holds for any $\mathfrak{A}$ with $Cl(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$: $(\mathfrak{A}_0, \tau_\alpha)$ is compact $\Longleftrightarrow$ (X, τ) is compact.*

Proof: According to lemma 7 we only must show that $(\mathfrak{A}_0, \tau_\alpha)$ is compact, if (X, τ) is compact. So, assuming (X, τ) to be compact, by proposition 6 every weakly relative complete subset of X is compact, too, and we have $\alpha \subseteq K(X)$. Now we will use Alexander's lemma: let $\underline{U}$ be a cover of $\mathfrak{A}_0$, consisting of subbase elements $K_i^{+\mathfrak{A}_0}$, $G_j^{-\mathfrak{A}_0}$ with K_i compact and G_j open.

$A := X \setminus \left(\bigcup\{G | G^{-\mathfrak{A}_0} \in \underline{U}\}\right)$ is closed.

By construction, $A \notin G^{-\mathfrak{A}_0}$ for any $G^{-\mathfrak{A}_0} \in \underline{U}$, so for $A \neq \emptyset$ there must exist some $K_0^{+\mathfrak{A}_0} \in \underline{U}$ with $A \in K_0^{+\mathfrak{A}_0}$, yielding that $K_0 \subseteq \bigcup\{G | G^{-\mathfrak{A}_0} \in \underline{U}\}$; K_0 compact $\Rightarrow \exists G_1, ..., G_n \in \underline{U}$ with $K_0 \subseteq \bigcup_{k=1}^{n} G_k$, but then $\{K_0^{+\mathfrak{A}_0}\} \cup \{G_1^{-\mathfrak{A}_0}, ..., G_n^{-\mathfrak{A}_0}\}$ is a cover of $\mathfrak{A}_0$.

If $A = \emptyset$, then $\bigcup\{G_i | G_i^{-\mathfrak{A}_0} \in \underline{U}\} = X$, so from the compactness of X the existence of some $G_1^{-\mathfrak{A}_0}, ..., G_n^{-\mathfrak{A}_0} \in \underline{U}$ with $X = \bigcup_{k=1}^{n} G_k$ follows. By lemma 2 then $\bigcup_{k=1}^{n} G_k^{-\mathfrak{A}_0} = \mathfrak{A}_0$ holds. □

Remark: Obviously, weak relative compactness is not the "weakest" condition, which one may require for the members of α in order to get compactness for $(\mathfrak{A}_0, \tau_\alpha)$ in this way. The only thing, we have to ensure, is that all members of α are compact, if X is compact. But, weak relative completeness ist a quite "weak" propertie as well, and it occurs in a natural manner by regarding some generalizations of uniform structures. It seems to be a sometimes useful common generalization of compactness, closedness and, in a (generalized) uniform setting, completeness.

COROLLARY 9. *Theorem 8 holds for each Δ-topology; especially it holds for the Vietoris topology, where the theorem was proved by Michael [4]. It also holds for the Fell topology $\tau_F = \tau_\alpha$, where $\alpha := \{K \subseteq X | \emptyset \neq K$ compact and closed $\} \subseteq Cl(X)$.*

COROLLARY 10. *Theorem 8 holds for the Fell topology $\tau_F = \tau_\alpha$, where $\alpha := K(X)$ and $\emptyset \in K(X)$.*

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